INCORPORATION OF SPACETIME SYMMETRIES
IN EINSTEIN'S FIELD EQUATIONS

Elias Zafiris

Theoretical Physics Group

Imperial College

The Blackett Laboratory

London SW7 2BZ

U.K.

e.mail:e.zafiris@ic.ac.uk

Abstract

In the search for exact solutions to Einstein's field equations the main simplification tool is the introduction of spacetime symmetries. Motivated by this fact we develop a method to write the field equations for general matter in a form that fully incorporates the character of the symmetry. The method is being expressed in a covariant formalism using the framework of a double congruence. The basic notion on which it is based is that of the geometrisation of a general symmetry. As a special application of our general method we consider the case of a spacelike conformal Killing vector field on the spacetime manifold regarding special types of matter fields. New perspectives in General Relativity are discussed.

1 Introduction

In recent years there has been a lot of research work in symmetries in General Relativity. Originally, the motivation was the need to simplify Einstein's field equations in the search for exact solutions, and the introduction of symmetries or collineations served as the basic tool.

1

The types of symmetries dealt with are those which arise from the existence of a Lie algebra of vector fields on the spacetime manifold which are invariant vector fields of certain geometrical objects on this manifold. The symmetries can be expressed in relations of the form $L_\xi W = Y$ where W and Y are two geometrical objects on the spacetime manifold and ξ^a is the vector field generating the symmetry [1].

The most important and common symmetries are those for which W and Y are one of the fundamental tensor fields of Riemannian geometry, namely

$$g_{ab}, \quad \Gamma^a{}_{bc}, \quad R_{ab}, \quad R_{abcd}$$

A diagram defining these symmetries and giving their relative hierarchy is given in [2].

The most extensively studied symmetries in the context of General Relativity are the Killing vectors and the Conformal Killing vectors [3,4,5,6,7]. Some work has also been done on Affine collineations [8,9,10], Curvature collineations [2,11,12] and contracted Ricci collineations [13,14,15]. The latter symmetries are much more difficult to be studied, because besides the metric they involve other geometrical objects which obey concrete conditions, that make the hadling and the interpretation of the equations difficult. A unified approach to all these symmetries from a purely differential geometric point of view has been recently given in [16].

It seems to us that although the motivation to study all the above mentioned symmetries was the simplification of Einstein's equations in the search for exact solutions, this initial aim has been partially unfulfilled. More concretely there doesn't seem to appear in the literature, so far as we know, a general framework which permits the incorporation of a symmetry of every possible kind in Einstein's equations for general matter.

Towards this direction we develop a method to write the field equations for

general matter in a form that fully incorporates the character of the symmetry. Our method is based on the notion of geometrisation of a general symmetry, that is we employ the description of it as necessary and sufficient conditions on the geometry of the integral lines of the vector field which generates the symmetry. Attempts towards this direction have been done for timelike Conformal Killing vectors [17,18], using the theory of timelike congruences [19,20,21,22,23], for spacelike Conformal Killing vectors [24], using the theory of spacelike congruences [25,26], and recently a step towards a generalisation of this idea has been presented in [27].

The most important aspect of the "geometrisation" method to study a symmetry is that the symmetry is expressed in a form that it is suited to the simplification of the field equations in a direct and inherent way.

The method we develop may be outlined as follows:

The introduction of a symmetry is most conveniently studied if we consider the Lie derivative of the field equations with respect to the vector ξ^a which generates the symmetry. After doing this we obtain an expression which on the left-hand side contains the Lie derivative of the Ricci tensor and on the right-hand side contains the Lie derivative of the energy momentum tensor. Eventually, we obtain the field equations as Lie derivatives along the symmetry vector of the dynamical variables. The $L_\xi R_{ab}$ can be calculated directly from the symmetry in terms of $\xi_{a;b}$. If we employ the corresponding theory of congruences we can express the $\xi_{a;b}$ in terms of the kinematical quantities (expansion, vorticity, shear) which characterise the congruence generated by the symmetry vector. The next step is to use the expression of a general symmetry as an equivalent set of conditions on the kimematical quantities characterising the congruence, namely we apply the geometrisation of a symmetry. If we substitute these conditions into the general expression for $L_\xi R_{ab}$ we manage to write directly the field equations, for any type

of matter, in a way that they inherit the symmetry of ξ_a.

In this work we compute the $L_\xi T_{ab}$ using the most general form of the energy-momentum tensor T_{ab}, and the method is being demonstrated assuming only that ξ_a is a spacelike vector orthogonal to the 4-velocity u^a of the observers. This choice is justified by the fact that we wish to keep the length of the equations as short as possible and at the same time exhibit all the steps of the method in a transparent manner, without falling into irrelevent to our purposes complications. Although we work with a spacelike symmetry vector it is plausible that our approach can be applied to an arbitrary (non-null) ξ_a, tilted with respect to u^a. (We note that the case in which ξ_a is timelike is much more simpler).

Having considered general matter it is evident that all the studies referring to various simplified types of matter like perfect fluids, charged fluids, anisotropic fluids and so on, consist special cases of our general scheme and they offer the ground to check the validity of our results. Thus we finally manage to recover and extend the results of the current literature, for example those of references [28,29,30] and [31].

The structure of the paper is as follows:

In section 2 we introduce the double congruence covariant framework permitting the introduction of arbitraty reference frames. In section 3 we present the generic formulation of symmetries in General Relativity and then we study the geometrisation of spacetime symmetries. The key formal results of the paper are contained in section 4 where we construct the symmetries-incorporated Einstein's field equations for general matter. Section 5 shows how these results are used in the situation of a spacelike Conformal Killing vector symmetry discussing the perfect fluid case. Finally we summarize and conclude in section 6, discussing further avenues of research. The applications considered here are taken just far enough to demonstrate and provide a familiar context for the techniques devel-

oped in this paper. I expect to return in later papers to the new applications which these techniques make possible.

The notation will be the usual one. **M** will denote the spacetime manifold with metric **g** of Lorentz signature $(-,+,+,+)$, which is assumed to be smooth. The Riemann, and Ricci tensors are denoted by R_{abcd}, and R_{ab}, respectively, whilst a semi-colon denotes a covariant derivative, a comma a partial derivative and L is the Lie derivative. Round and square brackets will denote the usual symmetrisation and skew-symmetrisation of indices. Latin indices take the values 0,1,2,3 and units are used for which the speed of light and Einstein's gravitational constant are both unity.

2 The double congruence framework

In order to develop our method we are going to use the most general approach to the description of reference frames, namely the theory of congruences. This approach is explicitly general covariant at each its step, permitting use of abstract representation of tensor quantities.

2.1 Timelike Congruences

The notion of reference frames is considered in the scope of classical physics, namely in the assumption that the observation and measurement procedures do not disturb the spacetime geometry. This means that the reference bodies are test ones and the only limitation on their motion is due to the relativistic causality principle: since a reference body represents an idealisation of sets of measuring devices and local observers, its' points worldlines should be timelike. Therefore we shall assume that the motion of a reference body is described by a congruence of timelike worldlines.

It is also obviously possible to an observer to move together with a frame of

reference, locally or in a spacetime region (globally). When an object moves together with a reference frame, it is geometrically identified with the latter, so that the worldlines of its mass points form a congruence. Such a congruence presents a complete characterisation of the reference frame, and with any reference frame a conceptual object (thus a test one) of the above type can be associated, which is called the body of reference. Since it models a set of observers and their measuring devices, but no photons, the reference frame congruence should be timelike.

We conclude that in the spacetime region where such a reference frame is realised, we consider a congruence of integral curves of a unit timelike vector field which is naturally interpreted as a field of 4-velocities of local observers, or equivalently of the worldlines of particles forming a reference body. Every local observer represents such a test particle.

It is important to note that the congruence concept is essential because for the sake of regularity of the mathematical description of the frame, these lines have not to mutually intersect, and they must cover completely the spacetime region under consideration, so that at every world point one has to find one and only one line passing through it. The simplest way to describe a reference frame is to identify the congruence of local observers with a congruence of the time coordinate lines, which should be timelike [32].

Thus all the tensor quantities, as well as all the differential operators, defined on the spacetime manifold, can be projected onto the physical time direction of a frame of reference and its local three-space with the help of appropriate projectors.

Let us assume that u is a future pointing unit timelike vector field ($u^a u_a = -1$) representing the 4-velocity field of a family of test observers filling the spacetime or some open submanifold of it.

The observer-orhogonal decomposition or [1+3] decomposition of the tangent space, and, in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $v(u)$ along u and the spatial projection operator $h(u)$ onto the orthogonal local rest space, which may be identified with mixed second rank tensors acting by contraction

$$\delta^a{}_b = v(u)^a{}_b + h(u)^a{}_b \tag{1}$$

where

$$v(u)^a{}_b = -u^a u_b \tag{2}$$

$$h(u)^a{}_b = \delta^a{}_b + u^a u_b \tag{3}$$

These satisfy the usual orthogonal projection relations

$$v(u)^2 = v(u), \quad h(u)^2 = h(u) \tag{4}$$

and

$$v(u)h(u) = h(u)v(u) = 0 \tag{5}$$

If S is a general tensor, then the "measurement" of S by the observer congruence is the family of spatial tensor fields which result from the spatial projection of all possible contractions of S by any number of factors of u. For example, if S is a (1,1)-tensor, then its measurement

$$S^a{}_b \rightarrow (u^d u_c S^c{}_d, h(u)^a{}_c u^d S^c{}_d, h(u)^d{}_a u_c S^c{}_d, h(u)^a{}_c h(u)^d{}_b S^c{}_d) \tag{6}$$

results in a scalar field, a spatial vector field, a spatial 1-form and a spatial (1,1)-tensor field. It is exactly this family of fields which occur in the orthogonal decomposition of S with respect to the observer congruence

$$
\begin{aligned}
S^a{}_b &= [v(u)^a{}_c + h(u)^a{}_c][v(u)^d{}_b + h(u)^d{}_b]S^c{}_d \\
&= [u^d u_c S^c{}_d]u^a u_b + \ldots + h(u)^a{}_c h(u)^d{}_b S^c{}_d
\end{aligned} \tag{7}
$$

7

The reference frame generated by u^a is described in the theory of timelike congruences by the introduction of its kinematical quantities

$$\sigma_{ab}, \quad \omega_{ab}, \quad \theta, \quad \dot{u}_a$$

which are defined by the irreducible decomposition or "measurement" of the covariant derivative $u_{a;b}$ with respect to u^a [22,23]

$$u_{a;b} = \sigma_{ab} + \omega_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b \tag{8}$$

where $\dot{u}_a$ denotes the acceleration vector of reference frame; θ the volume rate of expansion scalar; σ_{ab} is the rate of shear tensor, with magnitude

$$\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$$

and ω_{ab} is the vorticity tensor. It is convenient to define a vorticity vector

$$\omega^a = \frac{1}{2}\eta^{abcd}\omega_{bc}u_d$$

denoting the angular velocity vector of frame of reference where η^{abcd} is the totally skew-symmetric permutation tensor.

We note that that an overdot over a kernel letter means derivation with respect to u_a, hence $\dot{u}_a = u_{a;b}u^b$.

These concepts were borrowed by the references frames theory from hydrodynamics where they play an important role. They all are also important in the theory of null congruences, often used in the classification of gravitational fields by principal null directions (the Petrov types) and also generation of exact Einstein-Maxwell solutions [33].

Finally, we note that a partial splitting of spacetime based only on a timelike congruence (splitting off time alone) is referred as the congruence or "time plus space" or [1+3] decomposition, whereas a spacelike slicing of spacetime (splitting off space alone) is referred as the hypersurface splitting or "space plus time"

or [3+1] decomposition [34,35,36]. The two formalisms coincide in the case of the observer-orthogonal decomposition of the tangent space of the spacetime manifold.

If we had chosen the vector field generating a spacetime symmetry to be timelike, then the covariant formalism provided by the theory of timelike congruences would be enough to develop our method. Instead, we have chosen the more complicated case in which we have a spacelike symmetry vector. In this case a more sophisticated covariant formalism is needed and we are naturally led to use the concept of a double congruence.

2.2   Double Congruences

For our purposes we consider a double congruence which involves two vector fields: a timelike vector field u^a representing the 4-velocity of a family of test observers and a spacelike vector field ξ^a, which corresponds to a physical observable vector field, for example electric field or magnetic field. We demonstrate the previous point by considering the electromagnetic field strength tensor $F_{ab} = F_{[ab]}$.

Then

$$E^a = F^a{}_b u^b \quad and \quad H^a = \frac{1}{2} \eta^{abcd} u_b F_{cd}$$

Hence

$$E^a u_a = H^a u_a = 0$$

and both the electric and magnetic field as measured by the observers u^a have spacelike character.

We set $\xi^a = \xi \eta^a$, where η^a is a unit spacelike vector $\eta^a \eta_a = 1$ normal to the 4-velocity vector u^a.

To observe the given spacelike curves generated by η in the vicinity of the spacetime point P, we introduce, at that point, an observer moving with a 4-velocity u^a. We further suppose that the spacelike curve C is orthogonal to u^a

at the point P; then

$$u^a \eta_a = 0$$

It is important to emphasize that given a spacelike vector there is not a unique, orthogonal, timelike unit vector associated with it. We may add to the vector u^a any vector t^a such that

$$w^a = u^a + t^a$$

where t^a satisfies the conditions

$$t_a u^a = 0 \quad and \quad t_a t^a + 2 t_a u^a = 0$$

This freedom in our choice of an observer is essential in the covariant character of the theory.

In what follows we shall restrict attention to the observer moving with a 4-velocity u^a. Furthermore for the purpose of observation this observer erects a screen orthogonal to the spacelike curve C at P. That is, the congruence of curves passes perpendicularly through the screen.

Except at the given point P, the motions of the observers employed along the curve C have still to be specified. We require that the 4-velocities u^a of the observers used along C are related by:

$$p^a{}_c u_c{}^* = p^a{}_c \dot{\eta}^c \tag{9}$$

$$(u_a u^a)^* = 0, \quad (u_a \eta^a)^* = 0 \tag{10}$$

where an asterisk denotes derivation with respect to η^a, hence

$$u^{c*} = u_{a;b} \eta^b$$

The above ensures that u^a is always a unit vector orthogonal to η^a along C. Equations (14) and (15) are equivalent to the single condition

$$u_a{}^* = \dot{\eta}^a + (\dot{\eta}^b u^b) u^a - (\eta_b{}^* u^b) \eta^a \tag{11}$$

10

which we call the Greenberg's transport law for η^a [25].

The decomposition with respect to the double comgruence (u, η) or $[1+1+2]$ decomposition of the tangent space, and, in turn of the algebra of spacetime tensor fields, is accomplished by the temporal projection operator $v(u)$ along u, the spatial projection operator $s(\eta)$ along η, and the screen projection operator $p(u, \eta)$ which projects normally to both (u, η) onto an orthogonal two dimensional space, called the screen space.

All the above projection operators may be identified with mixed second rank tensors acting by contraction.

$$\delta^a{}_b = v(u)^a{}_b + s(\eta)^a{}_b + p(u, \eta)^a{}_b \tag{12}$$

$$v(u)^a{}_b = -u^a u_b \tag{13}$$

$$s(u)^a{}_b = \eta^a \eta_b \tag{14}$$

$$p(u, \eta)^a{}_b = \delta^a{}_b + u^a u_b - \eta^a \eta_b = h^a{}_b - \eta^a \eta_b \tag{15}$$

The covariant derivative $\eta_{a;b}$ can be decomposed with respect to the double congruence (u, η) as follows

$$\eta_{a;b} = A_{ab} + \eta_a{}^* \eta_b + u_a[\eta^t u_{t;b} + (\eta^t \dot{u}_t)u_b - (\eta^t u_t{}^*)\eta_b] \tag{16}$$

where

$$A_{ab} = p^c{}_a p^d{}_b \eta_{c;d} \tag{17}$$

We decompose A_{ab} further into its irreducible parts with respect to the orthogonal group:

$$A_{ab} = \mathcal{T}_{ab} + \mathcal{R}_{ab} + \frac{1}{2}\mathcal{E}p_{ab}(u, \eta) \tag{18}$$

where $\mathcal{T}_{ab} = \mathcal{T}_{ba}, \mathcal{T}^a{}_a = 0$ is the traceless part of A_{ab}, and $\mathcal{R}_{ab}$ is the rotation of A_{ab}. We have the relations

$$\mathcal{T}_{ab} = p^c{}_a p^d{}_b \eta_{(c;d)} - \frac{1}{2}\mathcal{E}p_{ab} \tag{19}$$

11

$$\mathcal{R}_{ab} = p^c{}_a p^d{}_b \eta_{[c;d]} \tag{20}$$

$$\mathcal{E} = p^{cd} \eta_{c;d} = \eta^a{}_{;a} + \dot{\eta}^a u_a \tag{21}$$

The tensors $\mathcal{R}_{ab}, \mathcal{T}_{ab}$ and the scalar $\mathcal{E}$ are defined as the kinematical quantities of the spacelike congruence and have the following physical significance: $\mathcal{R}_{ab}$ represents the screen rotation, $\mathcal{T}_{ab}$ the screen shear and $\mathcal{E}$ the screen expansion.

It is easy to show that in (16) the u^a term in parentheses can be written in a very useful form as follows:

$$-N_b + 2\omega_{tb}\eta^t + p^t{}_b \dot{\eta}_t$$

where

$$N_b = p^a{}_b(\dot{\eta}_a - u_a{}^*) \tag{22}$$

On using (22), equation (16) takes the form

$$\eta_{a;b} = A_{ab} + \eta_a{}^* \eta_b + p^c{}_b \dot{\eta}_c u^a + (2\omega_{tb}\eta^t - N_b)u_a \tag{23}$$

The vector N^a, which is called Greenberg's vector, is of fundamental importance in the theory of double congruences. Geometrically the condition $N^a = 0$ means that the congruences u^a and η^a are two surface forming. Kinematically, it means that the field η^a is "frozen in" along the observers u^a.

We show in this work that the role of Greenberg's vector is more general and establishes a connection between the field equations and the symmetries at kinematical level.

Using (16) we can also prove the following useful identities that the Lie derivatives of the projection tensors p_{ab} and h_{ab} obey and which we are going to use later

$$\frac{1}{\xi}L_\xi p_{ab} = 2[\mathcal{T}_{ab} + \frac{1}{2}\mathcal{E}p_{ab}] - 2u_{(a}N_{b)} \tag{24}$$

$$\frac{1}{\xi}L_\xi h_{ab} = 2[\mathcal{T}_{ab} + \frac{1}{2}\mathcal{E}p_{ab}] - 2u_{(a}N_{b)} + 2(\log\xi)_{(,a}\eta_{b)} + 2\eta^*_{(a}\eta_{b)} \tag{25}$$

12

3   Geometrisation of Spacetime Symmetries

The types of symmetries we are going to deal with, in what follows are those which arise from the existence of a Lie algebra of vector fields on the spacetime manifold which are invariant vector fields of certain geometrical objects on this manifold.

In Riemannian geometry the building block is the metric tensor g_{ab} in the sense that all the important geometrical objects of this geometry are expressed in terms of g_{ab}.

Following the standard literature [2,16,27], we define the generic form of a symmetry to be

$$L_\xi g_{ab} = 2\Psi g_{ab} + H_{ab} \tag{26}$$

where Ψ is a scalar field and H_{ab} is a symmetric traceless tensor field. Both of the fields satisfy a unique set of conditions specific to each particular symmetry and lead to a unique decomposssition of $L_\xi g_{ab}$. As special examples we mention that the Killing symmetries are characterized by

$$\Psi = H_{ab} = 0$$

the Homothetic symmetries by

$$\Psi = \phi = constant \quad and \quad H_{ab} = 0$$

the Conformal symmetries by

$$\Psi = \omega(x^a) \quad and \quad H_{ab} = 0$$

whereas the Affine symmetries by Ψ, H_{ab} such that

$$\Psi_{;c} = 0 \quad and \quad H_{ab;c} = 0$$

The generic form of a spacetime symmetry permits us to treat all of them in a unifying manner and is essential to our approach.

The geometrisation of spacetime symmetries is managed if we describe it as necessary and sufficient conditions on the geometry of the integral lines of the vector field which generates the symmetry. For our purposes we are going to study the geometrisation of a general spacetime symmetry generated by a spacelike vector field $\xi^a = \xi\eta^a$ using the framework of the double congruence (u, η) developed in section 2.

In the above framework the geometrisation of a a spacetime symmetry is established through the following theorem:

Theorem:The vector field $\xi^a = \xi\eta^a$ is a solution of

$$L_\xi g_{ab} = 2\psi g_{ab} + H_{ab}$$

if and only if

$$\mathcal{T}_{ab} = \frac{1}{2\xi}(p^c{}_a p^d{}_b - \frac{1}{2}p^{cd}p_{ab})H_{cd} \tag{27}$$

$$\dot{\eta}^a u_a = \frac{1}{\xi}(-\Psi) + \frac{1}{2\xi}H_{11} \tag{28}$$

$$\eta^{a*} = -u^a[-\dot{\log}\xi + \frac{1}{\xi}H_{21}] + p^{ab}[-(\log\xi)_{,b} + \frac{1}{\xi}H_{b2}] \tag{29}$$

$$\xi^* = \psi + \frac{1}{2}H_{22} \tag{30}$$

$$\mathcal{E} = \frac{2\psi}{\xi} + \frac{1}{2\xi}p^{ab}H_{ab} \tag{31}$$

$$N_a = -2\omega_{ab}\eta^b + \frac{1}{\xi}p^b{}_a H_{b1} \tag{32}$$

where we use the notational convention

$$Z\ldots_a\ldots u^a = Z\ldots_1\ldots \quad and \quad Z\ldots_a\ldots\eta^a = Z\ldots_2\ldots$$

for every tensor field Z.

Proof

14

The equation

$$L_\xi g_{ab} = 2\Psi g_{ab} + H_{ab}$$

can be written equivalently in the form

$$\xi(\eta_{a;b} + \eta_{b;a}) + \xi_{,a}\eta_b + \xi_{,b}\eta_a = 2\Psi g_{ab} + H_{ab} \qquad (33)$$

We decompose the above equation with respect to the double congruence (u, η). This can be done by contracting (26) with $u^a u^b$, $u^a \eta^b$, $u^a p^b{}_c$, $\eta^a \eta^b$, $\eta^a p^b{}_c$ and $p^a{}_c p^b{}_d$ respectively.

We obtain in turn:

$$u^a u^b: \quad \dot\eta^a u_a = \frac{1}{\xi}(-\Psi) + \frac{1}{2\xi}H_{11} \qquad (34)$$

$$u^a \eta^b: \quad \eta^{a*} u_a = -\dot{\log}\,\xi + \frac{1}{\xi}H_{21} \qquad (35)$$

$$u^a p^b{}_c: \quad \xi p^a{}_c(\dot\eta_a + \eta_{b;a}u^b) = p^a{}_c H_{a1} \qquad (36)$$

$$\eta^a \eta^b: \quad \xi^* = \Psi + \frac{1}{2}H_{22} \qquad (37)$$

$$\eta^a p^b{}_c: \quad p^b{}_c \eta^*_b = p^b{}_c[-(\log\xi)_{,b} + \frac{1}{\xi}H_{b2}] \qquad (38)$$

$$p^a{}_c p^b{}_d: \quad \mathcal{T}_{cd} + \frac{1}{2}\mathcal{E}p_{cd} = \frac{1}{2\xi}p^a{}_c p^b{}_d H_{ab} + \frac{1}{\xi}\Psi p_{cd} \qquad (39)$$

In equation (36) the first term of the lhs can be written in the form

$$\xi p^a{}_c(\dot\eta_a + \eta_{b;c}u^b) \quad =$$

$$\xi p^a{}_c(\dot\eta_a - u_a{}^* + u_{a;b}\eta^b - u_{b;a}\eta^b) \quad =$$

$$\xi N_c + \xi p^a{}_c(u_{a;b} - u_{b;a})\eta^b \quad =$$

$$\xi N_c + 2\xi p^a{}_c \omega_{ab}\eta^b \qquad (40)$$

Substituting (40) in (36) and using

$$p^b{}_a \omega_{bc}\eta^c = \omega_{ac}\eta^c$$

we obtain

$$N_a = -2\omega_{ab}\eta^b + \frac{1}{\xi}p^b{}_a H_{b1} \tag{41}$$

Equtions (35) and (38) give us the components of η^{a*} along u^a and on the screen space of η^a, u^a. It is possible to combine them into a single equation

$$\eta^{a*} = -u^a[-\dot{\log}\,\xi + \frac{1}{\xi}H_{21}] + p^{ab}[-(\log\xi)_{,b} + \frac{1}{\xi}H_{b2}] \tag{42}$$

Next we consider the trace and the traceless part of (39) and we obtain correspondingly

$$Trace: \quad \mathcal{E} = \frac{2\Psi}{\xi} + \frac{1}{2\xi}p^{ab}H_{ab} \tag{43}$$

$$Traceless \quad part: \quad \mathcal{T}_{ab} = \frac{1}{2\xi}(p^c{}_a p^d{}_b - \frac{1}{2}p^{cd}p_{ab})H_{cd} \tag{44}$$

The converse of the theorem is proved as follows:

We consider the tensor

$$\xi_{(a;b)} - \Psi g_{ab} - \frac{1}{2}H_{ab} = \xi\eta_{(a;b)} + \xi_{,(a}\eta_{b)} - \Psi g_{ab} - \frac{1}{2}H_{ab} \tag{45}$$

Contracting this with $u^a u^b$, $u^a \eta^b$, $u^a p^b{}_c$, $\eta^a \eta^b$, $\eta^a p^b{}_c$ and $p^a{}_c p^b{}_d$ respectively and applying relations (27)-(32), we prove that this tensor vanishes. The above completes the proof of the theorem.

The above theorem is of major importance to our approach because if we use it the symmetry is expressed in a form that it is possible to be incorporated in Einstein's field equations in a direct and inherent way.

Besides the theorem can be used in other ways depending on the information supplied. For example if information is available on the vector field ξ^a which generates the symmetry, one can investigate what type of symmetry the given vector field can generate. Conversely if information is available on the scalar and tensor fields ψ, H_{ab} one can use the theorem to obtain information on the vector field ξ^a which generates the symmetry.

16

4 The symmetries-incorporated Einstein's equations

The Einstein field equations with a non-zero cosmological constant can be written as

$$R_{ab} = T_{ab} + (\Lambda - \frac{T}{2})g_{ab} \tag{46}$$

where Λ is the cosmological constant and T is the trace of the energy-momentum tensor T_{ab}.

We wish to incorporate a general spacetime symmetry into Einstein's field equations. This is desirable because one can effectively eliminate the symmetry from the field equations and find solutions that will by construction comply with the symmetry.

The effects of the symmetries at the dynamical level are obtained by the Lie derivation of the Einstein field equations

$$L_\xi R_{ab} = L_\xi[T_{ab} + (\Lambda - \frac{T}{2})g_{ab}] \tag{47}$$

Eventually we obtain the field equations as Lie derivatives along the symmetry vector. We assume that the symmetry vector ξ^a is spacelike orthogonal to the 4-velocity u^a of the observers.

First of all we express the energy-momentum tensor T_{ab} in terms of its constituent dynamical variables by irreducibly decomposing it with respect to the double congruence (u, η). Next we compute the rhs of (47) in terms of the Lie derivatives of the dynamical variables. The next step is to compute the $L_\xi R_{ab}$ directly from the generic form of the symmetry in terms of $\xi_{a;b}$. Employing the (u, η) double congruence framework we use the expression of $\xi_{a;b}$ in terms of the kinematical quantities, namely we use the geometrisation of the symmetry applying the theorem proved in section 3. Thus we finally manage to write the

17

field equations in a way that they fully incorporate the character of a spacetime symmetry.

4.1 $[1+1+2]$ Irreducible decomposition of T_{ab}

It is well known from the literature that the irreducible decomposition of the energy momentum tensor T_{ab} with respect to the four velocity u^a of observers, defines the dynamical variables of spacetime [22,23]

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2 q_{(a} u_{b)} + \pi_{ab} \tag{48}$$

where μ and p denote the total energy density and the isotropic pressure, q_a is the heat flux vector, and π_{ab} is the traceless anisotropic stress tensor. These quantities include contributions by all sources, for example by an electromagnetic field. In particular q_a represents processes such as heat conduction and diffusion as well as the electromagnetic flux.

We can further decompose irreducibly the quantities q_a and π_{ab} using the double congruence framework, since -except the timelike congruence of observers- there is also defined a spacelike congruence on spacetime, representing covariantly a physical observable field as well as the character of the symmetry vector.

$$q^a = v \eta^a + Q^a \tag{49}$$

$$\pi_{ab} = \gamma(\eta_a \eta_b - \frac{1}{2} p_{ab}) + 2 P_{(a} \eta_{b)} + D_{ab} \tag{50}$$

where

$$v = q^a \eta_a, \quad Q^a = p^a{}_b q_b, \quad \gamma = \pi_{ab} \eta^a \eta^b$$

,

$$P_a = p^a{}_b \pi^b{}_c \eta^c, \quad D_{ab} = (p^c{}_a p^d{}_b - \frac{1}{2} p_{ab} p^{cd}) \pi_{cd}$$

The tensors Q^a, P^a, D_{ab} are on the screen space of (u, η) and D_{ab} is traceless.

The dynamical variables are constrained to obey the conservation equations

$$T^{ab}_{\ ;b} = 0 \tag{51}$$

which result from the identity $G^{ab}_{\ ;b} = 0$, where G_{ab} denotes the Einstein tensor. The above equations can be irrecucibly decomposed in the double congruence (u, η) framework into the following system of equations, which consist the conservation laws that the dynamical variables of spacetime have to satisfy.

$$\dot{\mu} + (\mu + p)\theta + q^a_{\ ;a} + q^a\dot{u}^a + \pi^{ab}\sigma_{ab} = 0 \tag{52}$$

$$(\mu + p)(\dot{u}_a\eta^a) + (p_{;c} + \dot{q}_c + \pi^b_{\ c;b})\eta^c + \theta(q^a\eta_a)$$
$$+q^b u_b^{\ *} - 2(p^c_{\ b}q^b)\omega_{bt}\eta^t = 0 \tag{53}$$

$$(\mu + p)\dot{u}_a p_{ac} + (p_;^{\ a} + \dot{q}^a + \pi^{ba}_{\ ;b})p_{ac}$$
$$+\theta q^a p_{ac} + q^b u^a_{\ ;b}p_{ac} = 0 \tag{54}$$

4.2 $[1+1+2]$ irreducible decomposition of the rhs of Einstein's field equations

4.2.1 Computation of $L_\xi R_{ab}$ using the field equations

We are going to compute the Lie derivative of the Ricci tensor using the field equations. The field equations for general matter (46) using (48) can be written in the form

$$R_{ab} = (\mu + p)u_a u_b + \frac{1}{2}(\mu - p + 2\Lambda)g_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \tag{55}$$

We study the case in which $\xi^a = \xi\eta^a$ corresponding to ξ^a spacelike and orthogonal to the 4-velocity of the observers. We decompose the above equation with respect to the double congruence (u, η). This is achieved by contracting the Lie derivative of the Ricci tensor with $u^a u^b$, $u^a\eta^b$, $u^a p^b_{\ c}$, $\eta^a\eta^b$, $\eta^a p^b_{\ c}$ and $p^a_{\ c}p^d_{\ b}$

respectively. If we denote by $\frac{1}{\xi}L_\xi R_{ab}[uu]$, $\frac{1}{\xi}L_\xi R_{ab}[u\eta]$, $\frac{1}{\xi}L_\xi R_{ab}[up]$, $\frac{1}{\xi}L_\xi R_{ab}[\eta\eta]$, $\frac{1}{\xi}L_\xi R_{ab}[\eta p]$, $\frac{1}{\xi}L_\xi R_{ab}[pp]$ the independent projections correspondingly we obtain:

$$
\begin{aligned}
\frac{1}{\xi}L_\xi R_{ab}[uu] &= [\frac{1}{2}(\mu + 3p)^* + (\mu + 3p - 2\Lambda)(\dot{u}^c\eta_c) \\
&- 2(q^c\eta_c)(\log\xi)\dot{} - 2(q^c N_c) + 2(q^c\eta_c)(u^{d^*}\eta_d)]u_a u_b \qquad (56)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\xi}L_\xi R_{ab}[u\eta] &= -2[-\frac{1}{2}(\mu - p + 2\Lambda)(u^{c^*}\eta_c) - (q^c\eta_c)^*\pi_{cd}\eta^c(u^{d^*} - \dot{\eta}^d) \\
&+ (q^c\eta_c)(\dot{\eta}^d u_d) + \frac{1}{2}(\mu - p + 2\Lambda + 2\pi_{cd}\eta^c\eta^d)(\log\xi)\dot{} \\
&- (q^c\eta_c)(\log\xi)^*]u_{(a}\eta_{b)} \qquad (57)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\xi}L_\xi R_{ab}[up] &= -2[\frac{1}{2}(\mu - p + 2\Lambda)N_d + (\mu + 3p - 2\Lambda)\omega_{cd}\eta^c \\
&- \pi_{ce}p^c{}_d(u^{e^*} - \dot{\eta}^e) - q_c A_{cd} - q^{c^*}p_{cd} \\
&+ p^c{}_d q_c(\dot{\eta}^e u_e) - (q^c\eta_c)p^e{}_d(\log\xi)_{;e} + \eta^e p^c{}_d\pi_{ec}(\log\xi)\dot{}]u_{(a}p^d{}_{b)} \quad (58)
\end{aligned}
$$

$$
\frac{1}{\xi}L_\xi R_{ab}[\eta\eta] = [\frac{1}{2}(\mu - p + 2\pi_{cd}\eta^c\eta^d)^* + (\mu - p + 2\Lambda + 2\pi_{cd}\eta^c\eta^d)(\log\xi)^*]\eta_a\eta_b \quad (59)
$$

$$
\begin{aligned}
\frac{1}{\xi}L_\xi R_{ab}[\eta p] &= 2[\frac{1}{2}(\mu - p + 2\Lambda)p_{cd}\eta^{c^*} + p^c{}_d(\pi_{ce}\eta^e)^* \\
&+ \pi^c{}_e\eta^e A_{cd} + 2(q_t\eta^t)\omega_{dc}\eta^c + \eta^c p^e{}_d\pi_{ce}(\log\xi)^* \\
&+ \frac{1}{2}(\mu - p + 2\Lambda + 2\pi_{ce}\eta^c\eta^e)p^f{}_d(\log\xi)_{,f}]\eta_{(a}p^d{}_{b)} \qquad (60)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\xi}L_\xi R_{ab}[pp] &= [(\mu - p + 2\Lambda)(\mathcal{T}_{cd} + \frac{1}{2}\mathcal{E}p_{cd}) + \frac{1}{2}(\mu - p)^*p_{cd} \\
&+ \pi^e{}_f(A_{ec}p_d{}^f + A_{ed}p_c{}^f) + \pi^{ef^*}p_{ec}p_{fd} \\
&+ 4q_c p^e{}_{(c}\omega_{d)f}\eta^f + 2\eta^e\pi_{ef}p_c{}^{(f}p^{t)}{}_d(\log\xi)_{,t}]p^c{}_{(a}p^d{}_{b)} \qquad (61)
\end{aligned}
$$

The above projections of the Lie derivative of the Ricci tensor have been obtained taking into account the irreducible decomposition of the symmetric $(0,2)$ tensor T_{ab} with respect to u^a. However, since ξ^a is spacelike we have further irreducible decompositions of the tensor fields. Concretely if we use equations (49) and (50) we express the Lie derivative of the Ricci tensor in terms of the

20

irreducible parts of the dynamical variables. Then the projections obtain the following irreducible form:

$$\frac{1}{\xi}L_\xi R_{ab}[uu] = [\frac{1}{2}(\mu + 3p)^* + (\mu + 3p - 2\Lambda)(\dot{u}^c\eta_c)$$
$$- 2Q^c N_c - 2v[(\log \dot{\xi}) + \eta^{c*}u_c]]u_a u_b \qquad (62)$$

$$\frac{1}{\xi}L_\xi R_{ab}[u\eta] = -2[\frac{1}{2}(\mu - p + 2\Lambda + 2\gamma)[(\log \dot{\xi}) + \eta^{c*}u_c]$$
$$+ P^c N_c - v^* - v[(\log \xi)^* - \dot{\eta}^c u_c]]u_{(a}\eta_{b)} \qquad (63)$$

$$\frac{1}{\xi}L_\xi R_{ab}[up] = -2[\frac{1}{2}(\mu - p + 2\Lambda - \gamma)N_c + (\mu + 3p - 2\Lambda)\omega_{dc}\eta^d$$
$$+ P_c[(\log \dot{\xi}) + \eta^{c*}u_c] + D_{dc}N^d - vp^d{}_c[\eta_d{}^* + \log \xi_{,d}]$$
$$+ Q_c(\dot{\eta}^d u_d) - Q_d A^d{}_c - p^d{}_c Q_d^*]p^c{}_{(a}u_{b)} \qquad (64)$$

$$\frac{1}{\xi}L_\xi R_{ab}[\eta\eta] = \frac{1}{2}[(\mu - p + 2\gamma)^* + (\mu - p + 2\Lambda + 2\gamma)(\log \xi)^*]\eta_a \eta_b \qquad (65)$$

$$\frac{1}{\xi}L_\xi R_{ab}[p\eta] = 2[p_{dc}P^* + \frac{1}{2}(\mu - p + 2\Lambda + 2\gamma)p^d{}_c[\eta_d{}^* + (\log \xi)_{,d}]$$
$$+ P^d[A_{dc} + (\log \xi)^* p_{cd}] - 2v\omega_{dc}\eta^d]p^c{}_{(a}\eta_{b)} \qquad (66)$$

$$\frac{1}{\xi}L_\xi R_{ab}[pp] = \frac{1}{2}[(\mu - p - \gamma)^* + (\mu - p + 2\Lambda - \gamma)\mathcal{E}]p_{ab}$$
$$+ p^c{}_a p^d{}_b D_{cd}{}^* + 2A_{c(b}D^c{}_{a)} + (\mu - p + 2\Lambda - \gamma)\mathcal{T}_{ab}$$
$$+ 4Q_{(a}\omega_{b)c}\eta^c + 2P_{(a}p^c{}_{b)}[\eta_c{}^* + (\log \xi)_{,c}] \qquad (67)$$

4.2.2 Incorporation of symmetries

Using the theorem proved in section 3 we reexpress the Lie derivative of the Ricci tensor in terms of the quantities Ψ, H_{ab} which characterise the generic form of a spacetime symmetry. Thus we finally obtain the independent projections of the Lie derivative of the Ricci tensor along the symmetry generating vector in terms of the fields characterising the symmetry Ψ, H_{ab} and the irreducible dynamical variables in the following form:

$$\frac{1}{\xi}L_\xi R_{ab}[uu] = [\frac{1}{2}(\mu + 3p)^* + (\mu + 3p - 2\Lambda)(\frac{\Psi}{\xi} - \frac{1}{2\xi}H_{11})$$
$$- \frac{2v}{\xi}H_{21} - 2Q^c N^c]u_a u_b \tag{68}$$

$$\frac{1}{\xi}L_\xi R_{ab}[u\eta] = -2[\frac{1}{2\xi}(\mu - p + 2\Lambda + 2\gamma)H_{21}$$
$$+ P_c N^c - v^* - v\frac{1}{2\xi}(4\Psi + H_{22} - H_{11})]u_{(a}\eta_{b)} \tag{69}$$

$$\frac{1}{\xi}L_\xi R_{ab}[\eta\eta] = [\frac{1}{2}[(\mu - p + 2\Lambda) + 2\gamma]^*$$
$$+ [(\mu - p + 2\Lambda) + 2\gamma](\log \xi)^*]\eta_a \eta_b \tag{70}$$

$$\frac{1}{\xi}L_\xi R_{ab}[pu] = -2[\frac{1}{2}[(\mu - p + 2\Lambda) + 2\gamma]N_c + (\mu + 3p - 2\Lambda)\omega_{tc}\eta^t - Q_t{}^* p^t{}_c$$
$$- \frac{v}{\xi}H_{t2}p^t{}_c - Q^t H_{tc} + D_{tc}N^t + \frac{1}{\xi}H_{21}P_c - Q^t \mathcal{R}_{tc}$$
$$- Q_c(\frac{2\Psi}{\xi} + \frac{1}{4\xi}p^{ef}H_{ef} - \frac{1}{2\xi}H_{11})]p^c{}_{(a}u_{b)} \tag{71}$$

$$\frac{1}{\xi}L_\xi R_{ab}[p\eta] = 2[p^d{}_c P_d{}^* + P_c(\frac{2\Psi}{\xi} + \frac{1}{4\xi}p^{ef}H_{ef} + \frac{1}{2\xi}H_{22}) + P_d H^d{}_c + P_d \mathcal{R}^d{}_c$$
$$+ \frac{1}{2\xi}[(\mu - p + 2\Lambda) + 2\gamma]p^d{}_c H_{d2} - 2v\eta^d\omega_{dc}]p^c{}_{(a}\eta_{b)} \tag{72}$$

$$\frac{1}{\xi}L_\xi R_{ab}[pp] = \frac{1}{2}(\mu - p - \gamma)^* p_{ab} + [(\mu - p + 2\Lambda) + 2\gamma][H_{ab}$$
$$+ \frac{1}{4\xi}p^{cd}H_{cd}p_{ab} + \frac{\Psi}{\xi}p_{ab}] + 2D_{c(a}\mathcal{R}^c{}_{b)} + \mathcal{E}D_{ab} + 2D_{c(a}H_{b)}{}^c$$
$$+ p^c{}_{(a}p^d{}_{b)}D_{cd}{}^* + \frac{2}{\xi}P_{(a}p^c{}_{b)}H_{c2} + 4Q_{(a}\omega_{b)t}\eta^t \tag{73}$$

4.3 [1 + 1 + 2] irreducible decomposition of the lhs of Einstein's field equations

We consider the generic form of a spacetime symmetry

$$L_\xi g_{ab} = 2\Psi g_{ab} + H_{ab}$$

where

$$H^a{}_a = 0 \quad and \quad H_{ab} = H_{ba}$$

The Lie derivative of the connection coefficients in terms of the Lie derivative of the metric tensor is expressed as follows [4]:

$$L_\xi \Gamma^a{}_{bc} = \frac{1}{2} g^{ad}[(L_\xi g_{ab})_{;c} + (L_\xi g_{ac})_{;b} - (L_\xi g_{bc})_{;d}] \tag{74}$$

Combining equations (26) and (74) we obtain directly:

$$L_\xi \Gamma^a{}_{bc} = \frac{1}{2} g^{ad}[2\Psi_{;c} g_{bd} + 2\Psi_{;b} g_{dc} - 2\Psi_{;d} g_b + H_{db;c} + H_{dc;b} + H_{bc;d}] \tag{75}$$

Equation (75) is equivalent to the following:

$$L_\xi \Gamma^a{}_{bc} = \Psi_{;c}\delta^a{}_b + \Psi_{;b}\delta^a{}_c - \Psi_{;}{}^a g^b{}_c + \frac{1}{2}[H_{db;c} + H_{dc;b} + H_{bc;d}] \tag{76}$$

Furthermore it is easy to obtain

$$L_\xi \Gamma^a{}_{ab} = 4\Psi_{;b} \tag{77}$$

In order to calculate the Lie derivative of the Ricci tensor we apply the relation

$$L_\xi R_{ab} = (L_\xi \Gamma^s{}_{ab}); s - (L_\xi \Gamma^s{}_{as}); b \tag{78}$$

Hence we obtain:

$$
\begin{aligned}
L_\xi R_{ab} &= [\Psi_{;a}\delta^s{}_b + \Psi_{;b}\delta^s{}_a - \Psi_{;}{}^s g_{ab} + \frac{1}{2}(H^s{}_{a;b} + H^s{}_{b;a} - H_{ab;}{}^s)]_{;s} - 4\Psi_{;a} \\
&= (\psi_{;ab} + \Psi_{b;a} - \psi_{;}{}^s{}_s g_{ab} - 4\Psi_{;ab}) + \frac{1}{2}(H^s{}_{a;bs} + H^s{}_{b;as} + H_{ab;}{}^s{}_s) \quad (79)
\end{aligned}
$$

Equivalently we obtain:

$$L_\xi R_{ab} = -2\Psi_{;ab} - \Box\psi g_{ab} + \frac{1}{2}(H^s{}_{a;bs} + H^s{}_{b;as} + H_{ab;}{}^s{}_s) \tag{80}$$

or

$$L_\xi R_{ab} = -2\Psi_{;ab} - \Box\psi g_{ab} + \frac{1}{2}\Lambda_{ab} \tag{81}$$

23

where

$$\Lambda_{ab} = H^s{}_{a;bs} + H^s{}_{b;as} + H_{ab;}{}^s{}_s \tag{82}$$

Next we decompose $\Psi_{;ab}$ with respect to the double congruence (u, η) framework into its irreducible parts

$$
\begin{aligned}
\Psi_{;ab} &= \lambda_\Psi u_a u_b - 2k_\Psi \eta_{(a} u_{b)} - 2s_{\Psi(a} u_{b)} \\
&+ \gamma_\Psi \eta_a \eta_b + 2p_{\Psi(a}\eta_{b)} + D_{\Psi ab} + \frac{1}{2}a_\Psi p_{ab}
\end{aligned} \tag{83}
$$

where

$$\lambda_\Psi = \Psi_{;ab}u^a u^b, \quad k_\Psi = \Psi_{;ab}\eta^{(a}u^{b)}, \quad s_{\Psi a} = p_a{}^b \Psi_{;bc}u^c$$

$$\gamma_\Psi = \Psi_{;ab}\eta^a\eta^b, \quad 2p_{\Psi a} = p_a{}^b \Psi_{;bc}\eta^c$$

$$D_{\Psi ab} = (p^c{}_a p^d{}_b - \frac{1}{2}p_{ab}p^{cd})\Psi_{;cd}, \quad a_\Psi = \Psi_{;ab}p^{ab}$$

Moreover we have

$$g_{ab}\Psi_{;ab} = -\lambda_\Psi + \gamma_\Psi + a_\Psi$$

Thus the sum $-2\Psi_{;ab} - \Box\psi g_{ab}$ is decomposed irreducibly as

$$
\begin{aligned}
-2\Psi_{;ab} - \Box\psi g_{ab} &= (-3\lambda_\Psi + \gamma_\Psi + a_\Psi)u_a u_b + 4k_\Psi \eta_{(a}u_{b)} \\
&+ 4s_{\Psi(a}u_{b)} + (\lambda_\Psi - 3\gamma_\Psi - a_\Psi)\eta_a\eta_b - 4p_{\Psi(a}\eta_{b)} \\
&- 2D_{\Psi ab} + (\lambda_\Psi - \gamma_\Psi - 2a_\Psi)p_{ab}
\end{aligned} \tag{84}
$$

Similarly the tensor Λ_{ab} defined by equation (82) is decomposed irreducibly as follows:

$$
\begin{aligned}
\frac{1}{2}\Lambda_{ab} &= \lambda_\Lambda u_a u_b - 2k_\Lambda \eta_{(a}u_{b)} - 2s_{\Lambda(a}u_{b)} \\
&+ \gamma_\Lambda \eta_a \eta_b + 2p_{\Lambda(a}\eta_{b)} + D_{\Lambda ab} + \frac{1}{2}a_\Lambda p_{ab}
\end{aligned} \tag{85}
$$

Thus the lhs of Einstein's equations is decomposed irreducibly as follows:

$$\frac{1}{\xi}L_\xi R_{ab}[uu] = \frac{1}{\xi}(\lambda_\Lambda - 3\lambda_\Psi + \gamma_\Psi + a_\Psi)u_a u_b \tag{86}$$

24

$$\frac{1}{\xi}L_\xi R_{ab}[u\eta] = \frac{1}{\xi}(-2(k_\Lambda - 2k_\Psi))\eta_{(a}u_{b)} \tag{87}$$

$$\frac{1}{\xi}L_\xi R_{ab}[up] = \frac{1}{\xi}[-2(s_{\Lambda c} - 2s_{\Psi c})]p^c{}_{(a}u_{b)} \tag{88}$$

$$\frac{1}{\xi}L_\xi R_{ab}[\eta\eta] = \frac{1}{\xi}[(\gamma_\Lambda - 3\gamma_\Psi) + (\lambda_\Psi - a_\Psi)]\eta_a\eta_b \tag{89}$$

$$\frac{1}{\xi}L_\xi R_{ab}[\eta p] = \frac{1}{\xi}[2(p_{\Lambda c} - 2p_{\Psi c})]p^c{}_{(a}\eta_{b)} \tag{90}$$

$$\frac{1}{\xi}L_\xi R_{ab}[pp] = \frac{1}{\xi}[D_{\Lambda ab} - 2D_{\Psi ab} + (\lambda_\Psi - \gamma_\Psi - 2a_\Psi + \frac{1}{2}a_\Lambda)p_{ab}] \tag{91}$$

4.4 The symmetries-incorporated field equations for the irreducible dynamical variables

Our purpose is to construct the field equations for general matter that the dynamical variables $\mu, p, \gamma, v, Q^a, P^a, D_{ab}$ of spacetime satisfy, in such a way that the information of any particular spacetime symmetry imposed, is directly incorporated in the form of the equations. As a first step we equate the results we have obtained previously for the rhs and the lhs of Einstein's equations:

$$[\tfrac{1}{2}(\mu + 3p)^* + (\mu + 3p - 2\Lambda)(\tfrac{\Psi}{\xi} - \tfrac{1}{2\xi}H_{11}) - \tfrac{2v}{\xi}H_{21} - 2Q^c N^c] =$$
$$\tfrac{1}{\xi}(\lambda_\Lambda - 3\lambda_\Psi + \gamma_\Psi + a_\Psi) \tag{92}$$

$$[\tfrac{1}{2}[(\mu - p + 2\Lambda) + 2\gamma]^* + [(\mu - p + 2\Lambda) + 2\gamma](\tfrac{\Psi}{\xi} + \tfrac{1}{2\xi}H_{22})] =$$
$$\tfrac{1}{\xi}((\gamma_\Lambda - 3\gamma_\Psi) + (\lambda_\Psi - a_\Psi)) \tag{93}$$

$$[\tfrac{1}{2\xi}(\mu - p + 2\Lambda + 2\gamma)H_{21} + P_c N^c - v^* - v\tfrac{1}{2\xi}(4\Psi + H_{22} - H_{11}) =$$
$$\tfrac{1}{\xi}(k_\Lambda - 2k_\Psi) \tag{94}$$

$$[\tfrac{1}{2}[(\mu - p + 2\Lambda) - \gamma]N_c + (\mu + 3p - 2\Lambda)\omega_{tc}\eta^t - Q_t{}^* p^t{}_c - \tfrac{v}{\xi}H_{t2}p^t{}_c - Q^t H_{tc}$$

$$+ D_{tc}N^t + \tfrac{1}{\xi}H_{21}P_c - Q^t \mathcal{R}_{tc} - Q_c(\tfrac{2\Psi}{\xi} + \tfrac{1}{4\xi}p^{ef}H_{ef} - \tfrac{1}{2\xi}H_{11})]$$

$$= \tfrac{1}{\xi}(s_{\Lambda c} - 2s_{\Psi c}) \tag{95}$$

$$[p^d{}_c P_d{}^* + P_c(\tfrac{2\Psi}{\xi} + \tfrac{1}{4\xi}p^{ef}H_{ef} + \tfrac{1}{2\xi}H_{22}) + P_d H^d{}_c + P_d \mathcal{R}^d{}_c$$

$$+ \tfrac{1}{2\xi}[(\mu - p + 2\Lambda) + 2\gamma]p^d{}_c H_{d2} - 2v\eta^d \omega_{dc}]$$

$$= \tfrac{1}{\xi}(p_{\Lambda c} - 2p_{\Psi c}) \tag{96}$$

$$(\mu - p - \gamma)^* + (\mu - p + 2\Lambda - \gamma)(\tfrac{1}{2\xi}p^{ef}H_{ef} + \tfrac{2\Psi}{\xi})$$

$$2D^{cd}H_{cd} + \tfrac{2}{\xi}P^c H_{c2} + 4Q^c \omega_{ct}\eta^t$$

$$= 2(\lambda_\Psi - \gamma_\Psi - 2a_\Psi + \tfrac{1}{2}a_\Lambda) \tag{97}$$

$$p^c{}_a p^d{}_b D_{cd}{}^* + \mathcal{E}D_{ab} + (\mu - p + 2\Lambda - \gamma)H_{ab} + 2D_{c(a}\mathcal{R}^c{}_{b)}$$

$$+ 2(p^c{}_{(a}p^d{}_{b)} - \tfrac{1}{2}p_{ab}p^{cd})(D_{ec}H^e{}_d + 2Q_{(c}\omega_{d)t}\eta^t)$$

$$= D_{\Lambda ab} - 2D_{\Psi ab} \tag{98}$$

The above system of equations can give us the irreducible equations that each of the dynamical variables of spacetime obey, if we manage to disentagle it through appropriate algebraic manipulations.

Equations (92),(93) and (97) can be written equivalently as follows correspondingly:

$$\mu^* + (\mu + \Lambda)(\tfrac{2\Psi}{\xi} - \tfrac{1}{\xi}H_{11}) + 3(p^* + (p - \Lambda))(\tfrac{2\Psi}{\xi} - \tfrac{1}{\xi}H_{11})$$

$$= \tfrac{2}{\xi}(\lambda_\Lambda - 3\lambda_\Psi + \gamma_\Psi + a_\Psi) + \tfrac{4v}{\xi}H_{21} + 4Q_c N^c \tag{99}$$

$$\mu^* + (\mu + \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{\xi}H_{22}) - [p^* + (p - \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{\xi}H_{22})]$$

$$+2[\gamma^* + \gamma(\tfrac{2\Psi}{\xi} - \tfrac{1}{\xi}H_{22})] = \tfrac{2}{\xi}[\lambda_\Psi - a_\Psi + (\gamma_\Lambda - 3\gamma_\Psi)] \tag{100}$$

$$\mu^* + (\mu + \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{2\xi}p^{ef}H_{ef}) - [p^* + (p - \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{2\xi}p^{ef}H_{ef})]$$

$$-[\gamma^* + \gamma(\tfrac{2\Psi}{\xi} + \tfrac{1}{2\xi}p^{ef}H_{ef})] = -4Q^c\omega_{ct}\eta^t$$

$$+\tfrac{2}{\xi}(\lambda_\Psi - \gamma_\Psi - 2a_\Psi + \tfrac{1}{2}a_\Lambda) - 2D^{cd}H_{cd} - \tfrac{2}{\xi}P^cH_{c2} \tag{101}$$

Moreover we note that

$$g^{ab}H_{ab} = 0 \quad or \quad p^{ab}H_{ab} = H_{11} - H_{22}$$

Hence equation (101) can be written in the form

$$\mu^* + (\mu + \Lambda)[\tfrac{2\Psi}{\xi} + \tfrac{1}{2\xi}(H_{11} - H_{22})] - [p^* + (p - \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{2\xi}(H_{11} - H_{22}))]$$

$$-[\gamma^* + \gamma(\tfrac{2\Psi}{\xi} + \tfrac{1}{2\xi}(H_{11} - H_{22}))] = -4Q^c\omega_{ct}\eta^t$$

$$+\tfrac{2}{\xi}(\lambda_\Psi - \gamma_\Psi - 2a_\Psi + \tfrac{1}{2}a_\Lambda) - 2D^{cd}H_{cd} - \tfrac{2}{\xi}P^cH_{c2} \tag{102}$$

Next if (102) is multiplied by a factor of two and added to (100) gives

$$\mu^* + (\mu + \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{3\xi}H_{11}) - (p^* + (p - \Lambda)(\tfrac{2\Psi}{\xi} + \tfrac{1}{3\xi}H_{11}))$$

$$+\gamma(\tfrac{1}{\xi}H_{22} - \tfrac{1}{3\xi}H_{11}) = \tfrac{1}{3\xi}(6\lambda_\Psi - 10a_\Psi + 2\gamma_\Lambda - 10\gamma_\Psi + 2a_\Lambda)$$

$$-\tfrac{8}{3}Q^c\omega_{ct}\eta^t - \tfrac{4}{3}D^{cd}H_{cd} - \tfrac{4}{3\xi}P^cH_{c2} \tag{103}$$

Furthermore we multiply (103) by a factor of three and we add to (99)

$$\mu^* + (\mu + \Lambda)\tfrac{2\Psi}{\xi} - (p - \Lambda)\tfrac{1}{\xi}H_{11} + \tfrac{1}{4\xi}\gamma(3H_{22} - H_{11})$$

$$= \tfrac{1}{2\xi}[\lambda_\Lambda + \gamma_\Lambda + a_\Lambda - 4\gamma_\Psi - 4a_\Psi] + \tfrac{v}{\xi}H_{21} + Q_cN^c$$

$$-2Q^c\omega_{ct}\eta^t - D^{cd}H_{cd} - \tfrac{1}{\xi}P^cH_{c2} \tag{104}$$

27

Next we consider the difference of (99) and (103)

$$p^* + (p - \Lambda)(\tfrac{2\Psi}{\xi} - \tfrac{2}{3\xi}H_{11}) - \tfrac{1}{4\xi}\gamma(H_{22} - \tfrac{1}{3}H_{11}) - \tfrac{1}{3\xi}(\mu + \Lambda)H_{11}$$

$$= \tfrac{1}{6\xi}(3\lambda_\Lambda - \gamma_\Lambda - a_\Lambda + 8a_\Psi + 8\gamma_\Psi - 12\lambda_\Psi) + Q_c N^c$$

$$+ \tfrac{2}{3}Q^c\omega_{ct}\eta^t + \tfrac{1}{3}D^{cd}H_{cd} + \tfrac{1}{3\xi}P^c H_{c2} + \tfrac{v}{\xi}H_{21} \tag{105}$$

Finally we subtract (102) from (100) and we obtain:

$$[\gamma^* + \gamma(\tfrac{2\Psi}{\xi} + \tfrac{1}{6\xi}H_{11} + \tfrac{1}{2\xi}H_{22})] + (\mu + \Lambda)\tfrac{1}{2\xi}(-\tfrac{1}{\xi}H_{11} + H_{22})$$

$$+ (p - \Lambda)\tfrac{1}{2\xi}(\tfrac{1}{\xi}H_{11} - H_{22}) = \tfrac{2}{3\xi}(\gamma_\Lambda - \tfrac{1}{2}a_\Lambda - 2\gamma_\Psi + a_\Psi)$$

$$+ \tfrac{2}{3}D^{cd}H_{cd} + \tfrac{2}{3}\tfrac{1}{\xi}P^c H_{c2} + \tfrac{4}{\xi}Q^c\omega_{ct}\eta^t \tag{106}$$

Thus the final irreducible form of the symmetries-incorporated Einstein field equations for the dynamical variables resulting from the decomposition of the energy-momentum tensor for general matter is described from the following system of equations:

Symmetries-incorporated field equations for μ, p, γ

$$\mu^* + (\mu + \Lambda)\tfrac{2\Psi}{\xi} - (p - \Lambda)\tfrac{1}{\xi}H_{11} + \tfrac{1}{4\xi}\gamma(3H_{22} - H_{11})$$

$$= \tfrac{1}{2\xi}[\lambda_\Lambda + \gamma_\Lambda + a_\Lambda - 4\gamma_\Psi - 4a_\Psi] + \tfrac{v}{\xi}H_{21} + Q_c N^c$$

$$- 2Q^c\omega_{ct}\eta^t - D^{cd}H_{cd} - \tfrac{1}{\xi}P^c H_{c2} \tag{107}$$

$$p^* + (p - \Lambda)(\tfrac{2\Psi}{\xi} - \tfrac{2}{3\xi}H_{11}) - \tfrac{1}{4\xi}\gamma(H_{22} - \tfrac{1}{3}H_{11}) - \tfrac{1}{3\xi}(\mu + \Lambda)H_{11}$$

$$= \tfrac{1}{6\xi}(3\lambda_\Lambda - \gamma_\Lambda - a_\Lambda + 8a_\Psi + 8\gamma_\Psi - 12\lambda_\Psi) + Q_c N^c$$

$$+ \tfrac{2}{3}Q^c\omega_{ct}\eta^t + \tfrac{1}{3}D^{cd}H_{cd} + \tfrac{1}{3\xi}P^c H_{c2} + \tfrac{v}{\xi}H_{21} \tag{108}$$

$$[\gamma^* + \gamma(\tfrac{2\Psi}{\xi} + \tfrac{1}{6\xi}H_{11} + \tfrac{1}{2\xi}H_{22}] + (\mu + \Lambda)\tfrac{1}{2\xi}(-\tfrac{1}{\xi}H_{11} + H_{22})$$

28

$$+(p-\Lambda)\tfrac{1}{2\xi}(\tfrac{1}{\xi}H_{11}-H_{22}) = \tfrac{2}{3\xi}(\gamma_\Lambda - \tfrac{1}{2}a_\Lambda - 2\gamma_\Psi + a_\Psi)$$

$$+\tfrac{2}{3}\tfrac{1}{\xi}P^c H_{c2} + \tfrac{4}{\xi}Q^c\omega_{ct}\eta^t + \tfrac{2}{3}D^{cd}H_{cd} \tag{109}$$

Symmetries-incorporated field equations for v, Q^a, P^a, D_{ab}

$$v^* + v\tfrac{1}{2\xi}(4\Psi + H_{22} - H_{11})$$

$$= [-\tfrac{1}{2\xi}(\mu - p + 2\Lambda + 2\gamma)]H_{21} + P_c N^c - \tfrac{1}{\xi}(k_\Lambda - 2k_\Psi) \tag{110}$$

$$Q_t^*{}^* p^t{}_c + Q^t H_{tc} + Q^t \mathcal{R}_{tc} + Q_c(\tfrac{2\Psi}{\xi} - \tfrac{1}{4\xi}(H_{11} + H_{22})]$$

$$= -\tfrac{1}{\xi}(s_{\Lambda c} - 2s_{\Psi c}) - [+\tfrac{1}{2}[(\mu - p + 2\Lambda) - \gamma]N_c + (\mu + 3p - 2\Lambda)\omega_{tc}\eta^t$$

$$-\tfrac{v}{\xi}H_{t2}p^t{}_c + D_{tc}N^t + \tfrac{1}{\xi}H_{21}P_c \tag{111}$$

$$p^d{}_c P_d^* + P_c[\tfrac{2\Psi}{\xi} + \tfrac{1}{4\xi}(H_{11} + H_{22})] + P_d H^d{}_c + P_d \mathcal{R}^d{}_c$$

$$= -\tfrac{1}{2\xi}[(\mu - p + 2\Lambda) + 2\gamma]p^d{}_c H_{d2} + 2v\eta^c\omega_{dc}$$

$$+\tfrac{1}{\xi}(p_{\Lambda c} - 2p_{\Psi c}) \tag{112}$$

$$p^c{}_a p^d{}_b D_{cd}^* + \mathcal{E}D_{ab} + 2D_{c(a}\mathcal{R}^c{}_{b)}$$

$$+2(p^c{}_{(a}p^d{}_{b)} - \tfrac{1}{2}p_{ab}p^{cd})D_{ec}H^e{}_d$$

$$= D_{\Lambda ab} - 2D_{\Psi ab} - (\mu - p + 2\Lambda - \gamma)H_{ab} - 4Q_{(c}\omega_{d)t}\eta^t(p^c{}_{(a}p^d{}_{b)} - \tfrac{1}{2}p_{ab}p^{cd}) \tag{113}$$

The system of equations (107)-(113) provide the desirable irreducible decomposition of Einstein's equations for general matter when an arbitrary symmetry has been introduced in spacetime, such that the information regarding the symmetry is explicitly contained in the field equations.

This system of equations provides the key formal results of the paper and generalises all the previous attempts to attack the problem, in an elegant and unifying manner. Moreover it greatly enlarges the scope of previous works since it can be applied to all types of symmetries as well as to all types of matter. Thus it consists the unified geometrical framework that we have to take into account when discussing Einstein's equations in spacetimes with general symmetries. Aside from elegance and covariance properties it provides the direct link among seperate approaches discussing particular symmetries and matter fields.

It is evident that the exact physical significance of the various parameters involved in the equations is provided by the special type of symmetry and matter that somebody considers applying the formalism. It is also expectable that the system of the symmetries-incorporated Einstein equations will reduce to familiar forms when applied to well-studied specific matter fields as well as symmetries. Even in this case there are some new insights to be gained from seeing these old calculations in the new general setting.

In the following section we demonstrate how these formal results are used in the case of a spacelike Conformal Killing vector symmetry, discussing the perfect fluid case. We expect to explore the applications of the symmetries-incorporated Einstein equations in the situation of Affine and Curvature collineations for various types of matter in later papers.

5 Application : The case of a spacelike conformal Killing vector symmetry

5.1 Kinematical Level

A spacelike conformal Killing vector $\xi^a = \xi\eta^a(\eta^a u_a = 0)$ satisfies the equation

$$L_\xi g_{ab} = 2\Psi g_{ab} \tag{114}$$

We note that in this case $H_{ab} = 0$.

From the theorem proved in section 3 the geometrisation of this particular symmetry is described by the following set of conditions:

$$\mathcal{T}_{ab} = 0 \tag{115}$$

$$\eta_a{}^* + (\log \xi)_{,a} = \frac{1}{2}\mathcal{E}\eta_a \tag{116}$$

$$\dot{\eta}_a u^a = -\frac{1}{2}\mathcal{E} \tag{117}$$

$$p^b{}_a(\dot{\eta}_b + u^t \eta_{t;b}) = 0 \tag{118}$$

The conformal factor Ψ reads

$$\Psi = \frac{1}{2}\xi\mathcal{E} = \xi^* \tag{119}$$

Due to the condition $u^a \eta_a = 0$ equation (118) can be written in the form

$$N_a = -2\omega_{ab}\eta^b \tag{120}$$

Moreover the decomposition of the covariant derivative of the spacelike vector field η_a is written as

$$\eta_{a;b} = A_{ab} + \eta_a{}^*\eta_b - \dot{\eta}_a u_b + p^c{}_b\dot{\eta}_c u_a \tag{121}$$

from which immediately obtain

$$L_\xi \eta^a = -\psi\eta^a \tag{122}$$

Moreover using (120) we easily find

$$L_\xi u^a = -\Psi u^a - \xi N^a \tag{123}$$

From (24) and (25) we can also compute

$$L_\xi p_{ab} = 2\Psi p_{ab} - 2\xi u_{(a}N_{b)} \tag{124}$$

$$L_\xi h_{ab} = 2\Psi h_{ab} - 2\xi u_{(a} N_{b)} \tag{125}$$

If $N^a = 0$ then (u, η) span a two dimensional surface in spacetime. The screen space is the orthogonal complement to this surface. The screen space admits ξ^a as a spacelike conformal Killing vector with conformal factor Ψ and its metric is defined by $p_{ab}(u, \eta)$. The rest space of the observers is the three dimensional space normal to u^a. The rest space admits admits ξ^a as a spacelike conformal Killing vector with conformal factor Ψ and its metric is defined by $h_{ab}(u)$.

Finally, for a conformal Killing vector we have from (80) the following further equations

$$L_\xi R_{ab} = -2\Psi_{;ab} - g_{ab} \Box \Psi \tag{126}$$

$$L_\xi R = -2\Psi R - 6\Box\Psi \tag{127}$$

5.2   Dynamical Level

We use the set of equations (107)-(113) in the special case we study and the results from the kinematical level. It is straightforward to obtain the following field equations:

$$\mu^* + (\mu + \Lambda)\mathcal{E} = -\frac{2}{\xi}(a_\Psi + \gamma_\Psi) + 2Q_a N^a \tag{128}$$

$$p^* + (p - \Lambda)\mathcal{E} = \frac{2}{3\xi}(2a_\Psi + 2\gamma_\Psi - 3\lambda_\Psi) + \frac{2}{3}Q_a N^a \tag{129}$$

$$\gamma^* + \gamma\mathcal{E} = -\frac{4}{3\xi}(\gamma_\Psi - \frac{1}{2}a_\Psi) - \frac{2}{3}Q_a N^a \tag{130}$$

$$v^* + v\mathcal{E} = \frac{2}{\xi}k_\Psi + P_a N^a \tag{131}$$

$$p^d{}_c Q^{d*} + \mathcal{E}Q_c = \frac{2}{\xi}s_{\Psi c} + (\mu + p - \frac{1}{2}\gamma)N_c + D_{dc} - Q_d \mathcal{R}^d{}_c \tag{132}$$

$$p^d{}_c P_d^* + \mathcal{E}P_c = -\frac{2}{\xi}P_{\Psi c} + vN_c - P_d \mathcal{R}^d{}_c \tag{133}$$

$$p^c{}_a p^d{}_b D_{cd}{}^* + \mathcal{E} D_{ab} = -\frac{2}{\xi} D_{\Psi ab} - 2\mathcal{R}_{c(a} D^c{}_{b)} + 2(p^c{}_a p^d{}_b - \tfrac{1}{2} p^{cd} p_{ab}) Q_{(c} N_{d)} \quad (134)$$

The dynamics is fully specified if in addition to the field equations we consider the conservation equations (52)-(54), which if expressed in terms of their ireducible parts read

$$\dot{\mu} + (\mu + p - \tfrac{\gamma}{2})\theta + v^* + 2v\mathcal{E} + p^{ab} Q_{a;b} + Q^a (\log \xi)_{;a}$$
$$+ 2Q^a \dot{u}_a + \tfrac{3}{2}\gamma (\dot{\log \xi}) + 2\sigma_{ab} P^a \eta^b + \sigma_{ab} D^{ab} = 0 \quad (135)$$

$$(p + \gamma)^* + \tfrac{1}{2}(\mu + p + 4\gamma)\mathcal{E} + \dot{v}$$
$$+ v[\theta + (\dot{\log \xi})] + p^{ab} P_{a;b} + P_a(\dot{u}^a - \eta^{a*}) = 0 \quad (136)$$

$$p^c{}_a[(\mu + p - \gamma/2)\dot{u}_c + p_{;c} + \dot{Q}_c + \tfrac{4}{3}\theta Q_c + \sigma_{cd} Q^d$$
$$+ 2v\sigma_{cd}\eta^d + P_c{}^* + \mathcal{R}_{cd} P^d + \mathcal{E} P_c + D_c{}^d{}_{;d} - \tfrac{3}{2}\gamma(\log \xi)_{;c} - \tfrac{1}{2}\gamma_{;c}] = 0 \quad (137)$$

The system of equations (128)-(137) characterise completely the dynamics induced by the Einstein equations when a spacelike Conformal Killing vector symmetry exists in spacetime. Thus it is possible if we specify a concrete type of matter field and after introducing appropriate systems of coordinates to obtain solutions which by construction comply with this symmetry.

Since all the dynamical information is contained in (128)-(137) all of the propositions or no-go theorems which can arise in the above setting, and gradually have appeared in the literature using various methods, are contained implicitly in the system we have constructed. In order to clarify this point, in what follows, we specialise our discussion in the case of a perfect fluid spacetime.

5.3 The Perfect fluid case

In the case of a perfect fluid the following relations hold:

$$\psi_{;ab} = \lambda_\Psi u_a u_b + \gamma_\Psi h_{ab} + \xi(\mu + p)N_{(a}u_{b)} \tag{138}$$

$$\Box\Psi = 3\gamma_\Psi - \lambda_\Psi \tag{139}$$

$$L_\xi R_{ab} = 3(\gamma_\Psi - \lambda_\Psi)u_a u_b$$
$$+(-5\gamma_\Psi + \lambda_\Psi)h_{ab} - 2\xi(\mu + p)N_{(a}u_{b)} \tag{140}$$

$$L_\xi R = -2\Psi R - 6(3\gamma_\Psi - \lambda_\Psi) \tag{141}$$

The general field equations in the case of a perfect fluid when a conformal Killing vector symmetry is introduced take the form:

$$\mu^* + (\mu + \Lambda)\mathcal{E} = -\frac{6}{\xi}\gamma_\Psi \tag{142}$$

$$p^* + (p - \Lambda)\mathcal{E} = \frac{2}{\xi}(2\gamma_\Psi - \lambda_\Psi) \tag{143}$$

$$(\mu + p)N_a = -\frac{2}{\xi}s_{\Psi a} \tag{144}$$

$$0 = k_\Psi = P_{\Psi a} = D_{\Psi ab} \tag{145}$$

$$a_\Psi = 2\gamma_\Psi \tag{146}$$

The consevation equations become

$$p^* + \frac{1}{2}(\mu + p)\mathcal{E} = 0 \tag{147}$$

$$(\mu - p + 2\Lambda\Psi = 4\gamma_\Psi + 2\lambda_\Psi \tag{148}$$

$$(\mu + p)p^b{}_a \dot{u}_b + p^c{}_a p_{,c} = 0 \tag{149}$$

Using the above sets of equations we can recover all the theorems that have been proved in the literature in a large series of publications using other methods,

quite easily. Since the large majority of these theorems are well known we are not going to state all of them, but we will restrict ourselves in mentioning two examples of propositions of this kind, so as to prove the increased flexibility of our formalism in familiar situations, and at the same time, to gain new insights by their embedding in a unified geometrical framework.

Proposition: Let ξ^a be a proper homothetic spacelike Killing vector orthogonal to the 4-velocity of observers of a perfect fluid spacetime ($\Psi_{;a} = 0$, $\Psi \neq 0$). Then $p - \Lambda = \mu + \Lambda$, namely matter is stiff and the current $j^a := \xi^{[a;b]}{}_{;b}$ vanishes.

Proof: If ξ^a is a conformal Killing vector we can easily obtain that the vector field $\Psi_{;a}$ obeys the following identity:

$$
\begin{aligned}
\Psi_{;a} &= -\frac{1}{3}(\mu + p)(u_t \xi^t)u_a + \frac{1}{6}(p - \mu - 2\Lambda)\xi^a \\
&\quad - \frac{2}{3}q_{(a}u_{b)}\xi^b - \frac{1}{3}\pi_{ab}\xi^b - \frac{1}{3}j^a
\end{aligned}
\tag{150}
$$

The above identity results if we apply the Ricci identity and use the expression of the Ricci tensor in terms of the dynamical variables from the field equations.

Since we refer to a perfect fluid spacetime the above equation reads

$$
(2\gamma_\Psi - \lambda_\Psi)\xi_a = \Psi(3\Psi_{;a} + j^a)
\tag{151}
$$

Next we use (143) and we immediately see that the condition $\Psi_{;a}$ implies that $p - \Lambda = \mu + \Lambda$, or else matter is stiff and the current $j^a := \xi^{[a;b]}{}_{;b}$ vanishes.

Proposition: Let ξ^a be a spacelike conformal Killing vector orthogonal to the 4-velocity of observers of a perfect fluid spacetime. Then if $\Lambda = 0$, $\mu + p = 0$ and the dominant energy condition holds, spacetime is a de Sitter spacetime with constant positive curvature and ξ^a reduces to a Killing vector.

Proof: From conservation equations we find that $\mu = const$ and from our assumptions positive since $\mu + p = 0$. Field equations (142) and (143) imply $\gamma_\Psi = -\lambda_\Psi$ and field equation (144) that $s_{\Psi a} = 0$. From these results we have:

$$
\Psi_{;ab} = \gamma_\Psi g_{ab}
\tag{152}
$$

$$L_\xi R_{ab} = -\gamma_\Psi g_{ab} \tag{153}$$

$$L_\xi R = -2\Psi R - 24\gamma - \Psi \tag{154}$$

From Einstein's field equations we find in this case:

$$R_{ab} = \mu g_{ab} \tag{155}$$

$$R = 4\mu = const \tag{156}$$

Thus spacetime is a de-Sitter spacetime. Moreover equation (155), (156) give

$$\mu\Psi = -3\gamma_\Psi \tag{157}$$

Applying Ricci's identity to the vector $\Psi_{;a}$ we find $R^{ab}\Psi_{;a} = 0$ which upon replacing R_{ab} from (155) we obtain $\mu\Psi_{;a} = 0$. But we have $\mu > 0$ hence the above gives $\Psi_{;a} = 0$ or $\gamma_\Psi = 0$. Thus from (157) we conclude that $\Psi = 0$, hence ξ^a reduces to a Killing vector.

6 Summary and Discussion

The basic tool to simplify Einstein's field equations, in search for exact solutions, has been the introduction of spacetime symmetries. The latter form Lie algebras of vector fields on the spacetime manifold which are invariant vector fields of certain geometrical objects on this manifold.

Related to the initial motivation, it would be very desirable to have in hand a general framework permitting the incorporation of a symmetry of every possible kind in Einstein's equations for general matter. There lacks in the literature a unified approach that will apply to all the symmetries and also to general matter. We have presented a general method to handle the field equations for general matter when a spacetime symmetry is introduced in its work.

Our method has been expressed in a covariant formalism, using the framework of a double congruence (u, η), permitting the introduction of arbitrary reference

frames and eventually systems of coordinates. The basic notion on which it is based is that of the geometrisation of a general symmetry, namely the description of it as an equivalent set of conditions on the kinematical quantities characterising the congruence of the integral lines of the vector field that generates the symmetry.

Using the above notion we finally manage to write the field equations which the dynamical variables obey for any type of matter, in a way that they inherit the symmetry of the generating vector.

The method has been applied in the case of a spacelike Conformal Killing vector symmetry recovering completely the existing literature. Further applications have been considered recovering results obtained by other methods in the case of a perfect fluid spacetime.

Thus we finally generalise and extend the results of the current literature in the form of the symmetries-incorporated Einstein's field equations for general matter.

The usefulness of such a construction lays on the following facts:

Firstly the field equations for general matter obtain an irreducible form with respect to the covariant congruence framework and specify completely the evolution of the spacetime dynamical variables inheriting, at the same time, the symmetry of the vector that generates it.

Secondly the system of the symmeties-incorporated Einstein equations give us the opportunity to eliminate the symmetry from the field equations trivially and find exact solutions that will by construction comply with the symmetry.

Thirdly, the set of the symmetries-incorporated field equations can be also used in another perspective equally significant. That is the above set of equations can be considered as a system of integrability conditions for the existence of a spacetime symmetry of a particular kind, in spacetime, when a concrete form of

the energy-momentum tensor is specified. Conversely it is possible imposing a spacetime symmetry to examine what types of matter can be present in spacetime. Related to the above remark the existing literature, using other methods, give us the information that some of the discussed symmetries are either absent or tightly restricted if a specific form of the energy momentum tensor is given. For example, affine and conformal vector fields cannot exist in vaccum spacetimes and affines are also forbidden when we have a perfect fluid with $0 \le p \ne \rho > 0$ and are also severely restricted for Einstein-Maxwell spacetimes [16,37]. Curvature symmetries are also heavily restricted in a similar way [38]. Along these lines research is in progress, using the formalism developed in this paper.

We finally wish to mention a short remark, which shows another fruitful direction in the same framework.

The method presented in this paper can be also used to study the symmetry inheritance of a kinematical or dynamical variable. This concept has been defined in the literature [29,31,39] to mean that in the presence of a spacetime symmetry a kinematical or dynamical variable X, satisfies an equation of the form

$$L_\xi X + k\psi X = 0$$

where k is a constant depending on the tensorial character of X. It is an appealing idea because it relates the symmetry with all the variables kinematical and dynamical making full use of the field equations.

Aknowledgements

References

[1] S.Kobayashi and K.Nomizu. "Foundations of Differential Geometry". Interscience, Vol. I, New York (1963)

[2] G.H.Katzin, J.Levine, and W.R.Davis, J.Math.Phys. **10**, 617 (1969)

[3] S.W.Hawking and G.F.R.Ellis, "The Large Scale Structure of Space-Time", Cambridge Univ.Press, Cambridge, UK, 1973.

[4] K.Yano, "The Theory of Lie Derivatives", North Holland, Amsterdam, 1955.

[5] C.W.Misner, K.S.Thorne, and J.A.Wheeler, "Gravitation", Freeman, San Francisco, 1973.

[6] L.Defrise-Carter, Comm.Math.Phys, **40**, 273 (1975)

[7] G.S.Hall, J.Math.Phys. **31**, 1198 (1990)

[8] R.Maartens, J.Math.Phys. **28**, 2051 (1987); A.A.Coley and B.O.J.Tupper, ibid. **31**, 649 (1990); S.Hojman and D.Nunez, ibid. **32**, 234 (1991)

[9] G.S.Hall and J.da Costa, J.Math.Phys. **29**, 2465 (1988)

[10] G.S.Hall, J.Math.Phys. **32**, 181 (1991)

[11] C.D.Collinson, J.Math.Phys. **11**, 818 (1970); P.C.Aichelburg, ibid **11**, 2458 (1970); R.A.Tello-Llanos, Gen.Rel.Grav. **20**, 765 (1988)

[12] G.S.Hall, Gen.Rel.Grav. **15**, 581 (1983)

[13] W.R.Davies, L.H.Green, and L.K.Norris, Nuovo Cimento B **34**, 256 (1976)

[14] M.Tsamparlis and D.P.Mason, J.Math.Phys. **31**, 1707 (1990)

[15] L.Nunez, U.Percoco, and V.M.Villalba, J.Math.Phys. **31**, 137 (1990)

[16] G.S.Hall, CBPF-MO-002/93

[17] J.Ehlers, P.Geren, and R.K.Sachs, J.Math.Phys. **9**, 1344 (1968)

[18] D.R.Oliver and W.R.Davis, Gen.Rel.Grav. **8**, 905 (1977)

[19] J.Ehlers, P.Jordan, W.Kundt, and R.Sachs, Akad.Wiss.Lit.Mainz Abh.Math.-Natur. Kl, **11**, (Mainz 4) 793 (1961)

[20] G.F.R. Ellis, J.Math.Phys. **8**, 1171 (1967)

[21] F.J.Greenberg, J.Math.Anal. and Applic. **29**, 645 (1970)

[22] G.F.R.Ellis, in "General Relativity and Cosmology: Proceedings of Course 47 of the International School of Physics Enrico Fermi" (R.Sachs, Ed.), Academic Press, New York, (1971)

[23] G.F.R.Ellis in "Cargese Lectures in Physics", Vol.6 (E.Schatzman, Ed.), Gordon and Breach, New York, 1973.

[24] D.P.Mason and M.Tsamparlis, J.Math.Phys. **26**, 2881 (1985)

[25] F.J.Greenberg, J.Math.Anal. and Applic. **30**, 128 (1970)

[26] D.P.Mason and M.Tsamparlis, J.Math.Phys. **24**, 1577 (1983)

[27] M.Tsamparlis, J.Math.Phys. **33**, 1472 (1992)

[28] D.P.Mason and M.Tsamparlis, J.Math.Phys. **31**, 1707 (1990)

[29] A.A.Coley and B.O.Tupper, J.Math.Phys. **30**, 2616 (1989)

[30] C.B.Collins, J.Math.Phys. **25**, 995 (1984)

[31] E.Saridakis and M.Tsamparlis, J.Math.Phys. **32**, 1541 (1991)

[32] Y.S.Vladimirov, N.Y.Mitskievich, and J.Horsky, "Space, Time, Gravitation", (Moscow:Mir) (1987)

[33] D.Kramer, H.Stephani, M.MacCallum, and E.Herlt "Exact Solutions of Einstein's Field Equations" (Cambridge, UK: Cambridge Univ. Press) (1980)

[34] R.Arnowitt, S.Deser, and C.W.Misner, in "Gravitation: An Introduction to Current Research" (L.Witten, Ed.), Wiley, New York, (1962)

[35] A.L.Zelmanov, Sov.Phys.Dokl. **18**, 231 (1973)

[36] A.E.Fischer and J.E.Marsden, in "General Relativity: An Einstein Centenary Survey" (S.W.Hawking and W.Israel, Eds.), Cambridge Univ.Press (1979)

[37] G.S.Hall, Gen.Rel.Grav. **20** 399 (1988)

[38] C.D.Collinson, and E.G.L.R.Vaz, Gen.Rel.Grav. **14**, 5 (1982)

[39] B.Carter and R.N.Henriksen, J.Math.Phys. **32**, 2580 (1991)